INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 8, 139–149 (1988)

A BOUNDARY ELEMENT FORMULATION FOR NATURAL CONVECTION PROBLEMS

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SUMMARY

This paper presents a boundary element formulation employing a penalty function technique for two-dimensional steady thermal convection problems. By regarding the convective and buoyancy force terms in Navier–Stokes equations as body forces, the standard elastostatics analysis can be extended to solve the Navier–Stokes equations. In a similar manner, the standard potential analysis is extended to solve the energy transport equation. Finally, some numerical results are included, for typical natural convection problems, in order to demonstrate the efficiency of the present method.

KEY WORDS Integral equations Boundary elements Natural convection Penalty function Navier–Stokes equations

INTRODUCTION

Thermal convection phenomena are of importance in many engineering and natural science problems. In particular, the buoyancy-driven flow of an incompressible viscous fluid under heating, the natural convection problem, is described by the Navier–Stokes and energy transport equations, consisting of a coupled set of non-linear equations. Because of difficulties in obtaining analytical solutions of the problem, several numerical formulations based on finite difference and finite element techniques have been developed and are widely used.[1,2]

More recently, procedures utilizing the boundary element method have also been derived for natural convection problems. These include velocity–pressure,[3,4] stream function–vorticity[5,6] and velocity–vorticity[7] formulations of the Navier–Stokes equations. Accurate solutions were obtained for classical problems such as the square cavity, for moderate Reynolds numbers ($Re \leqslant 100$) and Rayleigh numbers ($R_a \leqslant 10^4$).

The authors recently proposed a 'pseudo-body force' formulation for steady viscous flow problems.[8] Through the use of a penalty function technique, an analogy was made between the

0271–2091/88/020139–11$05.50

Received 28 January 1987

Navier–Stokes equations and the Navier equations of elastostatics. A boundary integral equation was then obtained, employing Kelvin's fundamental solution,[9] which considers the diffusive part of the process by means of boundary integrals alone. The convective terms, however, are regarded as pseudo-body forces and their influence computed iteratively by discretizing the domain of the problem into cells, similar to finite elements. The accurate evaluation of the convective terms plays a key role in the formulation, and the performance of different techniques for that end are compared in Reference 8.

This paper presents an extension of the pseudo-body force formulation to two-dimensional steady thermal convection problems. The Navier–Stokes equations are considered under the Boussinesq approximation and are formulated in terms of the primitive variables, velocity and pressure, although the pressure is later on eliminated through a penalty function technique.

Numerical results are presented for two typical natural convection problems, the square cavity flow and the flow between two horizontal, concentric cylinders. These results compare well with previous solutions obtained by using different numerical techniques.

THEORY

Basic equations

In this work, the flow is assumed to be steady, incompressible and two-dimensional. Boussinesq's approximation is employed; that is, the fluid is assumed to have constant properties except for the generation of buoyancy forces. If a Cartesian co-ordinate system is selected such that gravity forces work on the x_2 direction, the basic equations of natural convection are described by using a tensor notation as follows:

Continuity equation

$$v_{i,i} = 0; \tag{1}$$

Navier–Stokes equations

$$v_j v_{i,j} = -p_{,i} + Pr(v_{i,j} + v_{j,i})_{,j} + \delta_{2i} Ra Pr \theta; \tag{2}$$

Energy transport equation

$$v_j \theta_{,j} = \theta_{,jj}. \tag{3}$$

These equations were made dimensionless by choosing L (spacing between a hot wall (temperature T_H) and a cold wall (T_C)) and L^2/α(α, thermal diffusivity) as scale factors for length and time respectively. The temperature θ is expressed by $\theta = (T - T_C)/(T_H - T_C)$. Components of the velocity vector are represented by v_i; p, Pr and Ra indicate pressure, Prandtl number ($= \nu/\alpha$; ν, kinematic viscosity) and Rayleigh number ($= g\beta L^3(T_H - T_C)/\nu\alpha$; g, gravitational constant; β, coefficient of volumetric expansion) respectively. A comma stands for space partial derivatives—i.e., $v_{j,i} = \partial v_j / \partial x_i$—and δ_{ij} is the Kronecker delta symbol. When the same subscript appears twice in a term, the summation rule should be applied.

In order to evaluate the pressure term in equation (2), a penalty function technique is employed; that is, pressure is approximated as

$$p = -\lambda v_{i,i}, \tag{4}$$

where λ is a penalty parameter. Since the pressure p has a finite value, taking a value of λ which approaches infinity will make $v_{i,i}$ approach zero, enforcing in the limit the automatic satisfaction of the continuity equation (1). In the actual numerical calculations, a large but finite value is used for

λ, so the analysis is carried out taking a slight compressibility into consideration. By substituting equation (4) into equation (2), the following equations are obtained:

$$(\lambda + Pr)v_{j,ji} + Pr\, v_{i,jj} = v_j v_{i,j} - \delta_{2i} Ra\, Pr\, \theta. \tag{5}$$

On the other hand, under conditions of homogeneity and isotropy, the basic equation of elastostatics, which is the Navier equation, is as follows:

$$(\lambda' + \mu')u_{j,ji} + \mu' u_{i,jj} = -b_i, \tag{6}$$

where u_i and b_i stand for the components of displacement and body force vectors respectively; λ' and μ' are Lamé's constants.

On comparing equations (5) and (6), it will be noticed that they have similar left sides. Thus equation (5) can be treated analogously to a standard elastostatic problem, if the right-side terms are regarded as pseudo-body forces.

Furthermore, regarding the convective terms of the energy transport equation (3) as pseudo-source terms of a Poisson equation,

$$\theta_{,jj} = p, \tag{7}$$

an analogy can also be made between equation (3) and a standard potential analysis.

Boundary integral formulations

Initially, boundary integral formulations of equation (6) will be reviewed. The starting point is the weighted residual statement of equation (6); i.e.,

$$\int_\Omega [(\lambda' + \mu')u_{j,ji} + \mu' u_{i,jj} + b_i]u^*_{ki}\, d\Omega = 0. \tag{8}$$

The weighting functions u^*_{ki} are selected to satisfy Navier's equation in an infinite elastic medium with a discrete singularity, as follows:

$$(\lambda' + \mu')u^*_{kj,ji} + \mu' u^*_{ki,jj} + \delta_{ki}\delta(x, y) = 0, \tag{9}$$

where $\delta(x, y)$ is a Dirac delta function.

These fundamental solutions u^*_{kl} are known as Kelvin's solutions[9] and have the following form for plane strain problems:

$$u^*_{kl} = \frac{-1}{8\pi(1 - v')\mu'}[(3 - 4v')\ln r\,\delta_{kl} - r_{,k} r_{,l}]. \tag{10}$$

The tractions t^*_{kl} corresponding to u^*_{kl} are expressed by

$$t^*_{kl} = \frac{-1}{4\pi(1 - v')r}\left([(1 - 2v')\delta_{kl} + 2r_{,k} r_{,l}]\frac{\partial r}{\partial n} - (1 - 2v')(r_{,k} n_l - r_{,l} n_k)\right), \tag{11}$$

where $r = r(y, x)$ represents the distance between the load point y and the field point x, and n_k represents the direction cosines of the outward normal to the boundary of the body. v' indicates Poisson's ratio, which can be related to Lamé's constants as follows:

$$v' = \lambda'/2(\lambda' + \mu'). \tag{12}$$

Integrating by parts and applying Gauss's divergence theorem twice to equation (8), the following equations will be obtained:

$$C_{kl}(y)u_l(y)+\int_\Gamma t^*_{kl}(y,y')u_l(y')\,d\Gamma(y')$$
$$-\int_\Omega u^*_{kl}(y,y')t_l(y')\,d\Gamma(y')=\int_\Omega u^*_{kl}(y,x)b_l(x)\,d\Omega(x). \quad (13)$$

The coefficients $C_{kl}(y)$ can be determined by the position of the source point y. For example, $C_{kl}(y)=\delta_{kl}$ for $y\in\Omega$, $C_{kl}(y)=\delta_{kl}/2$ for $y\in\Gamma$ (the tangent plane at y is continuous in this case).

The boundary conditions of the problem are as follows:

$$u_i=\bar{u}_i \quad \text{on} \quad \Gamma_u,$$
$$t_i=\bar{t}_i \quad \text{on} \quad \Gamma_t=\Gamma-\Gamma_u, \quad (14)$$

where the bar indicates a prescribed value.

In standard elastostatic analysis, the boundary integral equations (13) are solved numerically for the boundary conditions (14) and known body forces b_i. In the present viscous flow analysis, b_i includes the unknown velocity vector v_i, its derivatives and the temperature θ. Therefore an iterative technique, which will be discussed in the next section, must be employed.

In order to evaluate the velocity derivatives of the convective terms at the internal points, the following boundary integral equations, which can be obtained for $y\in\Omega$ by differentiating equation (13) with respect to the co-ordinates of the source point y, were applied:

$$C_{kl}(y)u_{l,m}(y)+\int_\Gamma t^*_{klm}(y,y')u_l(y')\,d\Gamma(y')$$
$$-\int_\Gamma u^*_{klm}(y,y')t_l(y')\,d\Gamma(y')=\int_\Omega u^*_{klm}(y,x)b_l(x)\,d\Omega(x), \quad (15)$$

where

$$u^*_{klm}=\frac{\partial u^*_{kl}}{\partial x_m}, \quad t^*_{klm}=\frac{\partial t^*_{kl}}{\partial x_m}. \quad (16)$$

Similarly, a boundary integral equation equivalent to Poisson's equation (7) can be obtained as follows:

$$C(y)\theta(y)+\int_\Gamma q^*(y,y')\theta(y')\,d\Gamma(y')-\int_\Gamma \theta^*(y,y')q(y')\,d\Gamma(y')=-\int_\Omega \theta^*(y,x)p(x)\,d\Omega(x), \quad (17)$$

where $q=\partial\theta/\partial n$ is the normal flux and

$$\theta^*=\frac{1}{2\pi}\ln\frac{1}{r}, \quad q^*=\frac{\partial\theta^*}{\partial n}. \quad (18)$$

The temperature derivatives of the convective terms in the energy transport equation (3) are evaluated at the internal points by the following equations, which can be obtained by differentiating equation (17) with respect to the co-ordinates of the point y:

$$C(y)\theta_{,m}(y)+\int_\Gamma q^*_m(y,y')\theta(y')\,d\Gamma(y')$$
$$-\int_\Gamma \theta^*_m(y,y')q(y')\,d\Gamma(y')=-\int_\Omega \theta^*_m(y,x)p(x)\,d\Omega(x), \quad (19)$$

where

$$\theta_m^* = \frac{\partial \theta^*}{\partial x_m}, \quad q_m^* = \frac{\partial q^*}{\partial x_m}. \tag{20}$$

It should be noted that the pseudo-source term p includes not only the unknown temperature derivatives but also the velocities. This term, together with the buoyancy term in equation (5), couple the present equation with the Navier–Stokes equations. The iterative technique for solution of the coupled non-linear system of equations is described in the next section.

NUMERICAL IMPLEMENTATION

Numerical discretization

The numerical implementation of equation (13) will be described first. In order to solve the boundary integral equation (13), the boundary and domain of the region under study are divided into a number of small elements and cells respectively. Linear boundary elements and triangular linear internal cells were employed in this work. Following a standard collocation procedure,[8] a system of algebraic equations is obtained, which can be written in matrix form as

$$[\mathbf{H}]\{\mathbf{v}\} = [\mathbf{G}]\{\mathbf{t}\} + [\mathbf{C}]\{\mathbf{f}\}, \tag{21}$$

where $\{\mathbf{v}\}$ and $\{\mathbf{t}\}$ represent velocities and tractions at the boundary nodal points respectively, and $\{\mathbf{f}\}$ indicates the pseudo-body force terms at the internal nodal points; $[\mathbf{H}]$, $[\mathbf{G}]$ and $[\mathbf{C}]$ are influence matrices, the coefficients of which can be determined by integrating the fundamental solutions and interpolation functions over the boundary elements or the internal cells.

Taking the boundary conditions (14) into consideration, the matrix equation (21) can be modified into the following system of simultaneous equations

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{y}\} + [\mathbf{C}]\{\mathbf{f}\}. \tag{22}$$

If the pseudo-body force vector $\{\mathbf{f}\}$ is known, the unknown vector $\{\mathbf{x}\}$ can be obtained by a direct solution of system (22).

The velocity vector $\{\mathbf{v}\}$ and its derivative vector $\{\mathbf{dv}\}$ at internal nodal points are calculated by the following equations, obtained from boundary integral equations (13) and (15):

$$\{\mathbf{v}\} = [\mathbf{Af}]\{\mathbf{x}\} + \{\mathbf{yf}\} + [\mathbf{Cf}]\{\mathbf{f}\}, \tag{23}$$

$$\{\mathbf{dv}\} = [\mathbf{Adf}]\{\mathbf{x}\} + \{\mathbf{ydf}\} + [\mathbf{Cdf}]\{\mathbf{f}\}, \tag{24}$$

where $[\mathbf{Af}]$, $\{\mathbf{yf}\}$, $[\mathbf{Cf}]$, $\{\mathbf{Adf}\}$, $\{\mathbf{ydf}\}$ and $[\mathbf{Cdf}]$ represent known vectors and matrices calculated from geometric data and boundary conditions. These equations include not only the unknown vector $\{\mathbf{x}\}$ but also vector $\{\mathbf{f}\}$, which depends on the vectors $\{\mathbf{v}\}$ and $\{\mathbf{dv}\}$. Therefore iterative techniques must be employed in order to solve equation (22).

In a similar manner, the boundary integral equations (17) and (19) for the energy transport equation can be written in the following matrix form:

$$[\mathbf{At}]\{\mathbf{xt}\} = \{\mathbf{yt}\} + [\mathbf{Ct}]\{\mathbf{ft}\}, \tag{25}$$

$$\{\boldsymbol{\theta}\} = [\mathbf{Aft}]\{\mathbf{xt}\} + \{\mathbf{yft}\} + [\mathbf{Cft}]\{\mathbf{ft}\}, \tag{26}$$

$$\{\mathbf{d}\boldsymbol{\theta}\} = [\mathbf{Adft}]\{\mathbf{xt}\} + \{\mathbf{ydft}\} + [\mathbf{Cdft}]\{\mathbf{ft}\}, \tag{27}$$

where $\{\mathbf{xt}\}$ is the unknown value at the boundary nodal points, $\{\boldsymbol{\theta}\}$ is the temperature value at the internal nodal points, $\{\mathbf{d}\boldsymbol{\theta}\}$ are the derivatives of temperature at the internal nodal points, $\{\mathbf{ft}\}$ are the pseudo-source terms at the internal nodal points, and the other terms are known vectors and matrices calculated from geometric data and boundary conditions.

As $\{\mathbf{ft}\}$ depends on the vector $\{\mathbf{d\theta}\}$, iterative techniques must also be applied to solve equation (25).

Iterative techniques

In order to analyse thermal convection problems, equations (22) and (25) must be solved simultaneously. On the other hand, iterative techniques must be applied to solve each equation, as discussed in the previous subsection. Moreover, the pseudo-body force vector $\{\mathbf{f}\}$ includes the temperature vector $\{\boldsymbol{\theta}\}$, and the pseudo-source vector $\{\mathbf{ft}\}$ includes the velocity vector $\{\mathbf{v}\}$; that is, equations (22) and (25) are coupled with complex relations.

The iterative technique employed in this work comprises the following steps:

1. Assume values of velocities, their derivatives and the derivative of temperature at the internal nodal points, allowing vector $\{\mathbf{ft}\}$ to be calculated.
2. Solve the system of equations (25), evaluating the unknown vector $\{\mathbf{xt}\}$.
3. With vectors $\{\mathbf{ft}\}$ and $\{\mathbf{xt}\}$ obtained in the previous steps, compute the temperature $\{\boldsymbol{\theta}\}$ at the internal nodal points by using equation (26).
4. With vectors $\{\boldsymbol{\theta}\}, \{\mathbf{v}\}$ and $\{\mathbf{dv}\}$, calculate vector $\{\mathbf{f}\}$ at internal nodal points.
5. Solve the system of equations (22), evaluating the unknown vector $\{\mathbf{x}\}$.
6. With vectors $\{\mathbf{x}\}$ and $\{\mathbf{f}\}$, recompute the velocity vector $\{\mathbf{v}\}$ and its derivatives $\{\mathbf{dv}\}$ at the internal nodal points by using equations (23) and (24).
7. Examine the convergence of $\{\mathbf{v}\}$, $\{\mathbf{dv}\}$ and $\{\boldsymbol{\theta}\}$ at the internal nodal points.
8. If the results are not convergent, use the updated values of velocities, their derivatives and temperature at the internal nodal points to restart from the second step.

During the iteration steps, all matrices and some vectors which appear in equations (22)–(27) are kept constant. In particular, matrices $[\mathbf{A}]$ and $[\mathbf{At}]$ are constant during all the iterations, so their inversion is performed only once.

NUMERICAL RESULTS

Initially, the present formulation is applied to the square cavity flow problem, which is a typical benchmark problem in natural convection. Geometry and boundary conditions for temperature are shown in Figure 1. Non-slip boundary conditions (i.e., $u = v = 0$ along all the walls) are applied to the velocities. The discretization used is shown in Figure 2 and the corner points are treated as double nodes[9] in order to apply the boundary conditions for temperature.

The following results are compared in Table I with an accurate, benchmark FDM solution,[10] for Prandtl number $Pr = 0{\cdot}7$ and Rayleigh numbers $Ra = 10^3$ and 10^4:

(1) average Nusselt number $\overline{Nu} = \int_0^1 \partial\theta/\partial x|_{x=0}\,dy$;
(2) maximum and minimum local Nusselt numbers at the hot wall and their locations;
(3) maximum vertical velocity v_{max} on the horizontal mid-plane and its location;
(4) maximum horizontal velocity u_{max} on the vertical mid-plane and its location.

In addition, the flow and temperature fields are plotted in Figures 3 and 4 for $Ra = 10^3$ and 10^4 respectively. These results show a very good agreement with previously published numerical solutions.

One can notice from Figure 2 that the discretization employed is rather coarse. As a comparison, we mention that the FDM solution from Reference 10 utilized a 41×41 grid; i.e., 1681 nodal

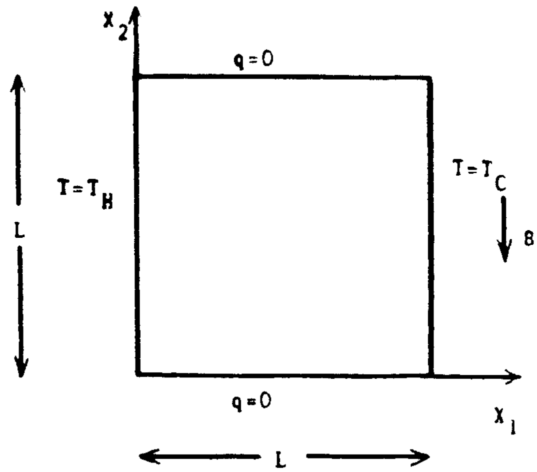


Figure 1. Geometry and temperature boundary conditions for square cavity

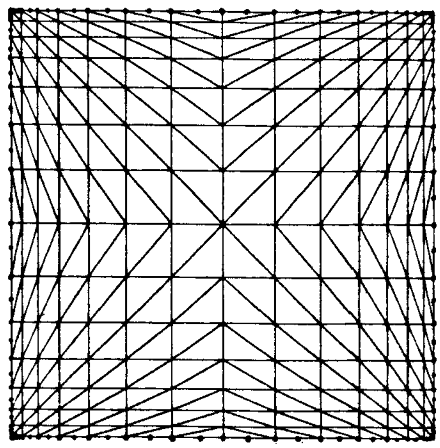
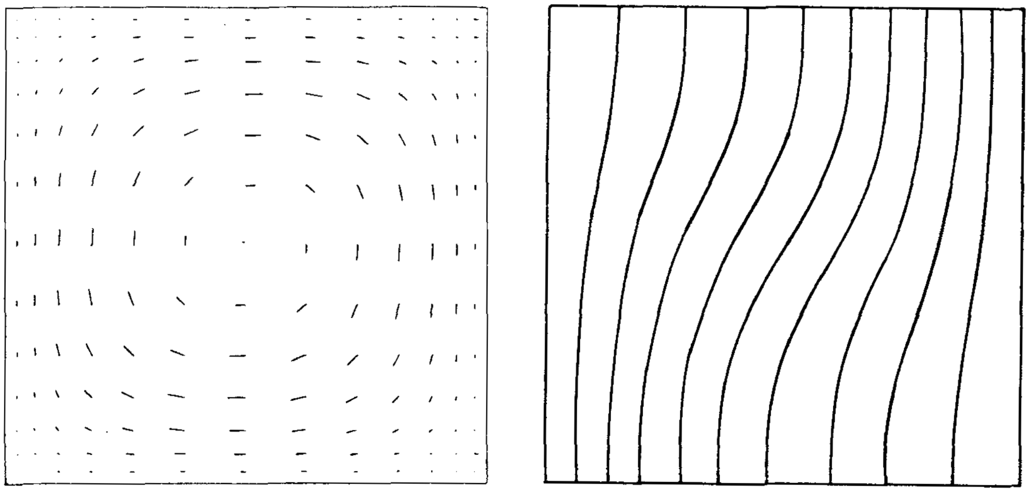


Figure 2. Discretization of square cavity (132 boundary nodes, 169 internal nodes)

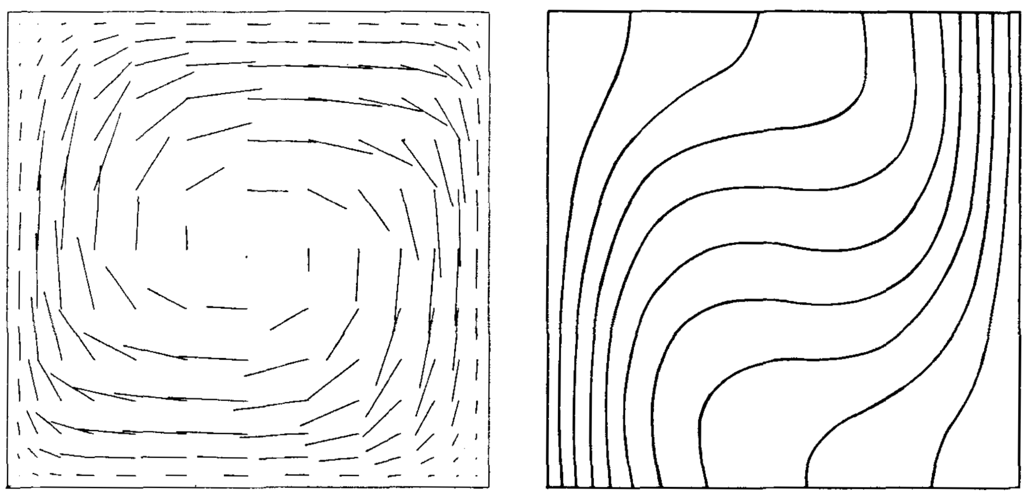
Table I. Results of the cavity flow problem

Ra	10^3		10^4	
	Present work	Reference 10	Present work	Reference 10
$\overline{Nu}$	1·114	1·118	2·219	2·238
Nu_{max}	1·489	1·506	3·484	3·527
y	0·088	0·086	0·147	0·143
Nu_{min}	0·697	0·691	0·610	0·586
y	1	1	1	1
u_{max}	3·654	3·657	15·399	16·178
y	0·819	0·814	0·819	0·823
v_{max}	3·703	3·702	19·782	19·643
x	0·181	0·178	0·113	0·119



(a) (b)

Figure 3. Results of the cavity flow problem ($Ra = 10^3$, $Pr = 0{\cdot}71$): (*a*) velocity distribution; (*b*) isothermal lines



(a) (b)

Figure 4. Results of the cavity flow problem ($Ra = 10^4$, $Pr = 0{\cdot}71$): (*a*) velocity distribution; (*b*) isothermal lines

points. The present discretization was also applied to the cavity problem for a Rayleigh number $Ra = 10^5$, but the solution failed to converge, indicating that more degrees of freedom are necessary for such a large Ra value.

Next, this formulation was applied to study the natural convection in the annulus between horizontal concentric cylinders, which is also a typical problem with a more complicated geometry. Non-slip conditions are also assumed and the boundary conditions for temperature are $T = T_H$ and $T_C (T_H > T_C)$ on the inner and outer cylinder walls respectively.

Taking symmetry into consideration, only half the domain needs to be discretized, as shown in Figure 5. The computer program developed treats symmetry by a reflection and condensation

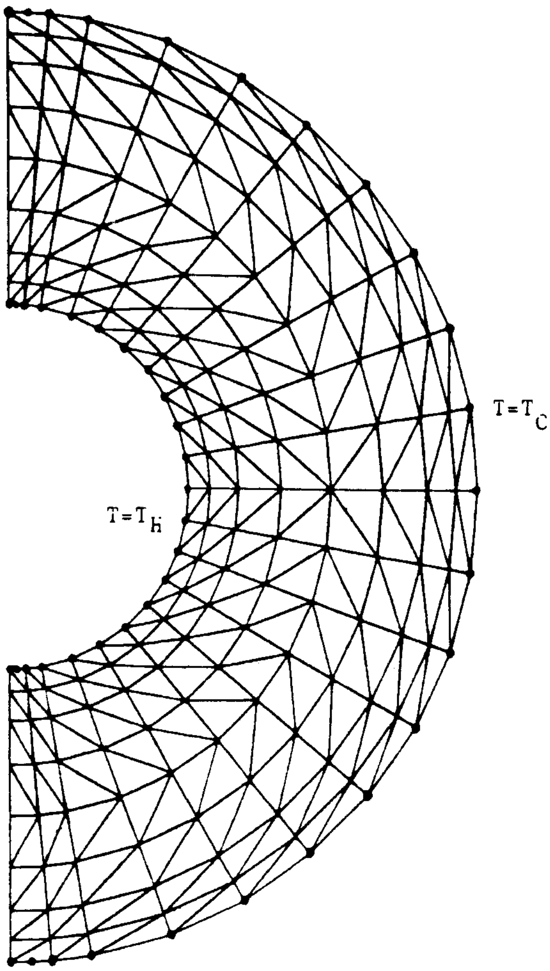


Figure 5. Discretization and temperature boundary conditions for concentric cylinders (46 boundary nodes, 147 internal nodes)

technique, such that no discretization of the symmetry axes is necessary.[9] Finally, the flow and temperature fields are shown in Figure 6 for $Ra = 10^3$, $Pr = 0{\cdot}7$ and $L/D_i = 0{\cdot}8$ (D_i, diameter of inner cylinder). These results are also in good agreement with previously published solutions.[11]

All calculations were carried out by using a value of 10^5 for the penalty parameter λ. In order to achieve convergence, an under-relaxation technique was employed in the solution of the system of algebraic equations.

CONCLUSIONS

A boundary element formulation with a penalty function technique was developed for steady thermal convection and applied to typical natural convection problems; that is, the square cavity flow and the flow between concentric cylinders. The results obtained were in close agreement with previously published numerical solutions. The present formulation is now being extended to deal with three-dimensional natural convection problems, the results of which will be reported in a future paper.

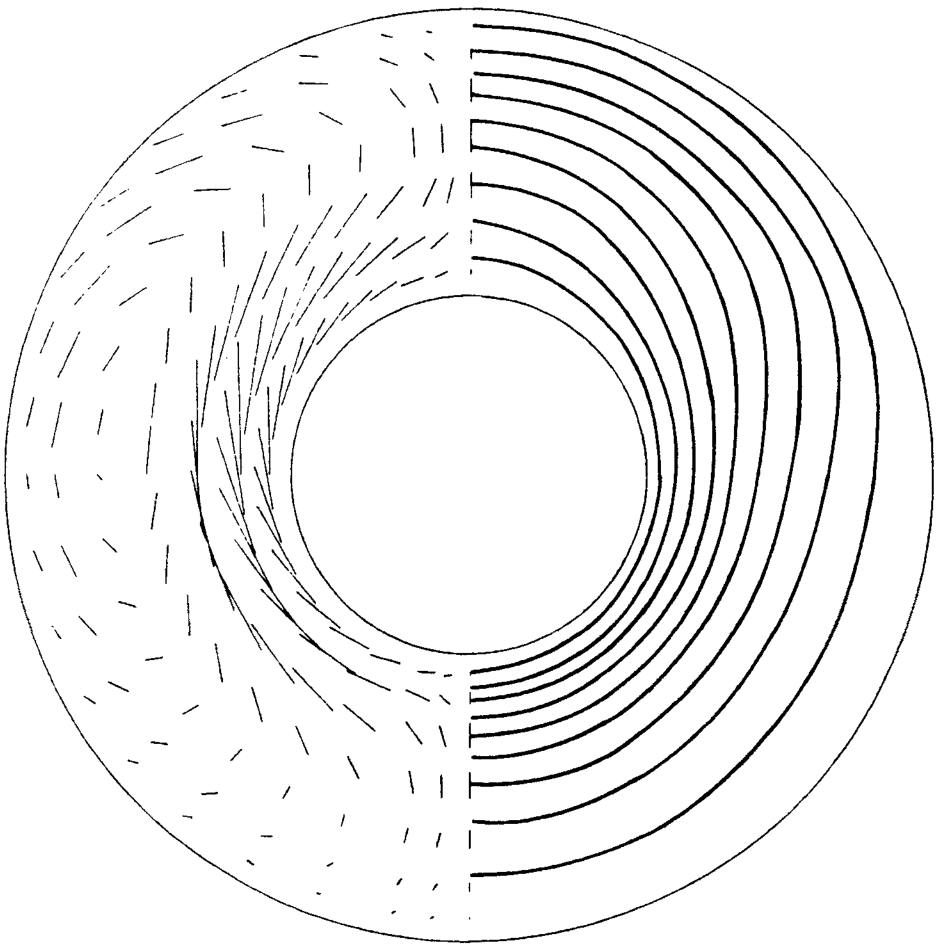


Figure 6. Velocity distribution and isothermal lines for flow between concentric cylinders ($Ra = 10^3, Pr = 0{\cdot}7, L/D_i = 0{\cdot}8$)

ACKNOWLEDGEMENTS

This work was developed while two of the authors (K. Kitagawa and L. C. Wrobel) were visiting researchers at the Computational Mechanics Institute, Southampton, U.K. Financial support from the Toshiba Corporation, Japan and CAPES/ME, Brazil is gratefully acknowledged.

REFERENCES

1. R. W. Lewis and K. Morgan (eds), *Numerical Methods in Thermal Problems, Vol. 4*, Pineridge Press, Swansea, 1985.
2. C. Taylor *et al.* (eds), *Numerical Methods in Laminar and Turbulent Flow, Vol. 4*, Pineridge Press, Swansea, 1985.
3. T. Kuroki, K. Onishi and N. Tosaka, 'Thermal fluid flow with velocity formulation using boundary elements and penalty function methods', in C. A. Brebbia and G. Maier (eds), *Boundary Elements VII*, Springer, Berlin, 1985, pp. 2/107–2/114.
4. N. Tosaka and N. Fukushima, 'Integral equation analysis of laminar natural convection problems', in M. Tanaka and C. A. Brebbia (eds), *Boundary Elements VIII*, Springer, Berlin, 1986, pp. 803–812.
5. K. Onishi, T. Kuroki and M. Tanaka, 'An application of a boundary element method to natural convection', *Appl. Math. Modelling*, **8**, 383–390 (1984).
6. K. Onishi, T. Kuroki and M. Tanaka, 'Boundary element method for laminar viscous flow and convective diffusion problems', in C. A. Brebbia (ed.), *Topics in Boundary Element Research, Vol. 2*, Springer, Berlin, 1985, pp. 209–229.

7. P. Skerget, A. Alujevic and C. A. Brebbia, 'BEM for laminar motion of isochoric viscous fluid', in J. J. Connor and C. A. Brebbia (eds), *BETECH 86*, CM Publications, 1986, pp. 397–419.
8. K. Kitagawa, C. A. Brebbia, L. C. Wrobel and M. Tanaka, 'Boundary element analysis of viscous flow by penalty function formulation', *Eng. Anal.*, **3**, 194–200 (1986).
9. C. A. Brebbia, J. C. F. Telles and L. C. Wrobel, *Boundary Element Techniques*, Springer, Berlin, 1984.
10. G. de V. Davis and I. P. Jones, 'Natural convection in a square cavity: a comparison exercise', in R. W. Lewis *et al.* (eds), *Numerical Methods in Thermal Problems, Vol. 2*, Pineridge Press, Swansea, 1981, pp. 552–572.
11. T. H. Kuehn and R. J. Goldstein, 'An experimental and theoretical study of natural convection in the annulus between horizontal concentric cylinders', *J. Fluid Mech.*, **74**, 695–719 (1976).